TEXTBOOK

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МЕДИЦИНСКАЯ И БИОЛОГИЧЕСКАЯ ФИЗИКА

Рекомендовано Министерством общего и профессионального образования Российской Федерации в качестве учебника для студентов медицинских специальностей высших учебных заведений

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Chapter 2
Elements of probability theory

The probability theory studies objective laws related to random events, random variables, and random processes. Physicians rarely think that making diagnosis has a probabilistic character. As it was wittily observed, only a post mortem examination can reliably determine the diagnosis of a person who has died.

§ 2.1. EXPERIMENT WITH MULTIPLE OUTCOMES. RANDOM EVENT

The probability theory studies laws inherent to experiments *with multiple outcomes*. This is a term for experiments, whose results are not possible to *foresee accurately*. E.g., when someone plays roulette, the ball thrown on the rotating wheel can stop in *any* of 37 numbered slots (0, 1, ..., 36), but until the wheel stops the slot number remains unknown.

Experiment and its outcomes

The concepts of "*experiment*" and "*outcome*" are the primary concepts of probability theory.

***An experiment* is a sequence of actions to be done under certain conditions.**

***An outcome* means what is *directly* obtained because of the experiment.**

Experiment is determined if the conditions of the experimentation are specified and the set *of all* its possible *outcomes* is known; the latter is denoted by the letter Ω. For example, for playing *roulette*, the croupier winds the game wheel round, throws the ball on it, waits for the wheel to stop and announces the number of the slot which the ball is located in. The foregoing actions are a description of an *experiment*. *The experiment outcome* is the announced number of the slot. The set of *all* possible *outcomes* consists of 37 numbers: $\Omega = \{0, 1, 2, ..., 36\}$.

Note that because in each experiment there appears **only one** of all possible outcomes.

In medical research, an experiment is any examination of a patient, e.g., determination of glucose content in his or her blood taken from the vein. The *outcome* is the result of examination.

Random event

Individual outcomes of experiment, as a rule, do not have independent significance. Some of their *sets*, which are called *events*, are of practical interest. E.g., a roulette player can bet his or her money on "even". He *wins* if the ball stops in the slot with an *even* number, and *loses* otherwise. The *specific* number of a slot does not matter. In this case, there are two events of practical interest: "win" is getting an even number, and "loss" is getting an odd number. Nothing else matters.

Outcomes of medical research are also grouped into significant events. E.g., 3 events are considered for determining blood glucose content: this index is *normal* (3.9–6.4 mmol/L), *below normal*, *above normal*. But the specific value of the index (e.g., 5.18 mmol/L) does not have practical importance. In this example, the event "normal" is the set of all numbers within the interval (3.9–6.4 mmol/L).

***A random event* or simply *an event* is a set of experiment outcomes with a practical interest. Such outcomes are called *conducive* to this event (or *favorable* for it).**

An event occurs if the result of experiment is one of *favorable outcomes*.

In probability theory, capital Latin letters (A, B, C...) denote random events.

§ 2.2. OPERATIONS ON EVENTS. OPPOSITE EVENT. INCOMPATIBLE EVENTS

In order to explain *what this event is*, it is necessary to enumerate all possible outcomes of the experiment (Ω) and designate *favorable* ones. In some cases, it is simple to do, and in other cases, it is much more difficult.

For example, in the experiment the shooter is to fire *one* shot at a target. In this case, only *two* outcomes are possible: A (hit) or B (miss). These outcomes are the simplest events.

Now consider an experiment when the shooter fires *two* shots at the target. In this case, *four elementary* outcomes are possible:

1) A_1 and A_2 — two hits;
2) A_1 and B_2 — a hit and a miss;
3) B_1 and A_2 — a miss and a hit;
4) B_1 and B_2 — two misses.

The event C consisting in the fact that the target is hit by two shots is favored by three outcomes, in which there is at least one hit:

$$C = \{(A_1 \text{ and } A_2), (A_1 \text{ and } B_2), (B_1 \text{ and } A_2)\}.$$

To describe *complex* events, they are presented as a result of operations on simpler events. Such operations are *addition* and *product* of events.

The *sum*, or *union*, of two events A and B is the event that is occurrence of at least one of them.

The *sum* of events is denoted as follows: $\boldsymbol{A+B}$. (In some textbooks, the *sum* of events is denoted as $A \bigcup B$.)

The event $\boldsymbol{A+B}$ is the set of outcomes which are favorable to *at least one* of events A, B.

***The product*, or *intersection*, of two events A and B is called an event consisting in occurrence of both events.**

The *product* of events is denoted as follows: $\boldsymbol{A \cdot B}$. (In some textbooks *intersection* of events is denoted as $A \bigcap B$.)

The event $\boldsymbol{A \cdot B}$ represents a set of outcomes, favorable *for each* event (both for event A and for event B).

The foregoing complex event C, which is the hit on the target by two shots, is written as operations of addition and multiplication of simple events (A is the hit, B is the miss) in the following way:

$$C = A_1 \cdot A_2 + A_1 \cdot B_2 + B_1 \cdot A_2.$$

Let us analyze a simple example that explains the technique of performing operations of event addition and multiplication. A dice is thrown. Event A is even number falling: $A = \{2, 4, 6\}$. Event B is falling of a number that is a multiple of three: $B = \{3, 6\}$.

- **Addition:** $\boldsymbol{A+B}$ is a number that is **either** even **or** is divided by 3: $A + B = \{2, 3, 4, 6\}$.
- **Product:** $\boldsymbol{A \cdot B}$ is a number that is **both** even, **and** is divided by 3: $A \cdot B = \{6\}$.

It is convenient to illustrate operations on events graphically with special Venn diagrams. In them, the space of elementary outcomes Ω is designated with a circumference, whose points are interpreted as elementary outcomes. Simple events are designated by some figures, e.g., ovals. The image of the sum and the product of events is shown in fig. 2.1 (the dark area).

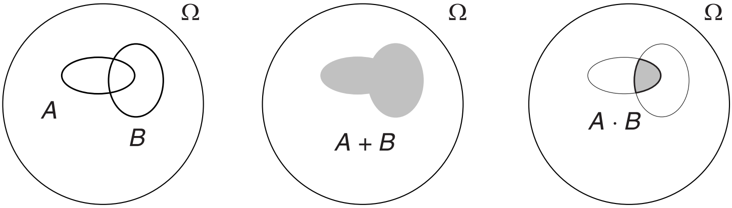

Fig. 2.1. Graphic representation of sum and product of two events

Opposite event

To each event A, it is possible to map the opposite event $\overline{A}$ (is read "not A"), consisting of all outcomes, *unfavorable* for A. A graphic illustration of events A and $\overline{A}$ is represented in fig. 2.2.

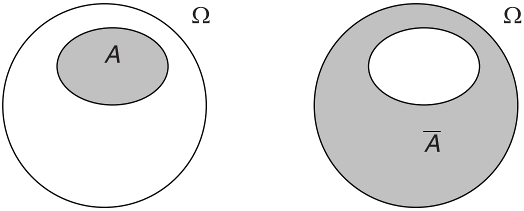

Fig. 2.2. Event A and event $\overline{A}$ which is opposite to the former

The event, which is *opposite* to event *A*, is the following: for experimentation, event *A did not occurr*.

Let us note that $A + \overline{A} = \Omega$.

Incompatible events

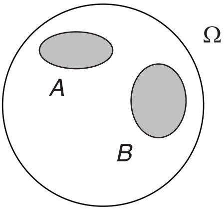

Fig. 2.3. Incompatible events do not have common outcomes

Incompatible events A are of great importance in the probability theory.

***Incompatible events* are events that cannot happen simultaneously (for one experimentation).**

Incompatible events *do not have a common outcome* that is why they are represented by non-intersecting figures (fig. 2.3).

An important special case of incompatible events is initial and opposite events (A and $\overline{A}$).

§ 2.3. CLASSIC DEFINITION OF PROBABILITY. AXIOMS OF PROBABILITY THEORY

It is possible to notice that for multiple experimentations with random outcomes some events occur more often than others. E.g., if you throw a dice many times, an even number will fall in about *half* of the cases, while the proportion of the numbers multiples of three will be approximately *one third*.

Probability of event

In order to compare random events by the degree of probability of their occurrence, it is necessary to associate a number with each of them, which is the greater the more possible this event is. This number determines the *probability* of the event.

***Probability of an event* is a quantitative characteristic of the possibility of its occurrence.**

Probability is denoted by the letter "P": the probability of event A is denoted by $P(A)$ or P_A.

The theory of probability was initially invented for analysis of games of chance and was applied to experiments, whose all outcomes are *equally possible*.

Outcomes of experiment are called *equally possible*, if no objective reasons by virtue of some outcomes can be more given than others.

E.g., due to the symmetry of a dice, the chances of all its faces falling are *equal*. Therefore, a throw of a dice is an experiment with equally possible outcomes.

Classical definition of probability

Let us consider an experiment with N equally possible outcomes. Let us denote the number of outcomes, which are favorable for event A, as N_A.

***The probability of a random event* is the ratio of the number of favorable outcomes for this event to the number of all equally possible outcomes of this experiment:**

$$P_A = N_A/N. \tag{2.1}$$

Historically, this formula was given the name "Classical Definition of Probability". It was the first quantitative result of a proposed theory which made it possible to determine probabilities of success in various kinds of games of chance. Let us consider the application of this definition to dice game.

Problem. Players A and B play by throwing two dices each. Player A wins when the sum of his or her points is seven. Player B wins when the sum of his or her points is eight. Who benefits from this game?

Solution. The outcome of each throw is falling of a *pair* of faces. Due to the symmetry of dices, all outcomes are equal, and their number is $N = 6 \cdot 6 = 36$.

The win of the player A (the event A) 6 is favored by six outcomes (1–6, 6–1, 2–5, 5–2, 3–4, 4–3); $N_A = 6$. The win of player B (the event B) 6 is favored by 5 outcomes (2–6, 6–2, 5–3, 3–5, 4–4); $N_B = 5$. Using formula (2.1), let us find $P_A = 6/36$, $P_B = 5/36$. Thus, player A benefits from this game.

Axioms of the probability theory

Not all experiments have equally possible outcomes. For instance, in shooting at a target, the possibilities of hit and miss are obviously different. In order to generalize the concept of probability to arbitrary experiments with random outcomes, it was necessary to introduce a number of general concepts and properties.

The limits within which probability of an event are changes established according to two special concepts.

1. A *certain event* is an event that is *sure* to occur because of an experiment. Such an event is the set of *all* possible outcomes Ω.

2. An *impossible event* is an event that in this experiment cannot occur at all. For instance, in playing roulette, number 38 cannot fall; it is simply not on the wheel. An impossible event is denoted with the symbol ∅.

The probability of a certain event is taken as *one*:

$$P_\Omega = 1.$$

The probability of an impossible event is taken as *zero*:

$$P(\varnothing) = 0.$$

Two more axioms are added to these properties of probability:

- the probability of any event A lies between zero and one:

$$0 \le P_A \le 1;$$

- the probability of the sum of *incompatible* events is equal to the sum of their probabilities:

$$P(A + B) = P_A + P_B. \quad (2.2)$$

It can be proved that the probability of the sum of *joint* events is given by the following formula:

$$P(A + B) = P_A + P_B - P(A \cdot B). \quad (2.3)$$

§ 2.4. RELATIVE FREQUENCY OF AN EVENT, THE LAW OF LARGE NUMBERS

Conditions in which it is permissible to use the classical definition of probability are extremely rare since experiments with *equally possible* outcomes are rather an exception than a rule. If the outcomes *are not equally possible*, then the probability of an event cannot be calculated by formula (2.1).

Let us consider a method of *experimental* evaluation of some event probability A. Let us reiterate the same experiment several times and count in how many experiments this event has *occurred*.

A relative frequency of a certain event A in a series of accomplished experiments is the ratio of the number of experiments (n_A), in which the event occurred, to the total number of accomplished experiments (n):

$$P_A^* = \frac{n_A}{n}. \tag{2.4}$$

If n is small, the relative frequency of an event is random at a certain extent. However, as the number of experiments increases, the frequency tends to *stabilize*, approaching, with negligible fluctuations, a certain constant. The table below shows how the frequencies (P^*) of falling of tails change upon an increase in the number of throws (n) for a symmetrical coin.

Table 2.1

n	10	50	75	100	200	300	400	500	600
P*	0.400	0.540	0.493	0.510	0.505	0.503	0.498	0.502	0.499

The plot curve corresponding to these changes is shown in fig. 2.4.

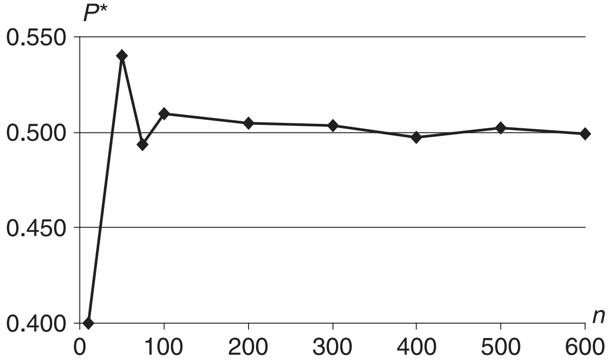

Fig. 2.4. Convergence of the relative frequency of the event to its probability

The relative frequency of an event and its probability are connected by the ***law of large numbers.***

As the number of experiments increases unlimitedly, the frequency of the event tends to its probability:

$$\frac{n_A}{n} \to P(A) \text{ with } n \to \infty. \tag{2.5}$$

This ratio is sometimes called the statistical probability definition. In accordance with the law of large numbers, the probability of an event can be taken as its relative frequency with a large number of experiments.

§ 2.5. INDEPENDENT EVENTS. ADDITION AND MULTIPLICATION OF PROBABILITIES OF INDEPENDENT EVENTS

The concept of *statistical independence* occupies an important place in the probability theory and is defined as follows:

Events *A* and *B* are called ***independent***, if the fact of occurrence of one of them does not change the probability of occurrence of the other.

A typical example of *independent* events is events that occur in experiments with *independent outcomes.*

Two experiments are called ***independent* if the outcome of one experiment cannot influence the outcome of the other.**

E.g., if you throw two dices, the result of the first throw does not affect the result of the second throw.

For independent events, the *theorem of multiplication of probabilities* is applicable.

The probability of an event, that is the product of independent events A and B, is equal to the product of their probabilities:

$$P(A \cdot B) = P_A \cdot P_B. \tag{2.6}$$

Example. Let there be five black and 10 white balls in one box, and three black and 17 white balls in the other box. The problem is to find the probability of drawing a black ball from each box simultaneously.

Event A is removing a black ball from the first box:

$$P_A = 5/15 = 1/3\ .$$

Event B is removing a black ball from the other box:

$$P_B = 3/20.$$

Event $A \cdot B$ is both balls being black:

$$P(A \cdot B) = P_A \cdot P_B = 1/3 \cdot 3/20 = 1/20.$$

Application of the probability multiplication theorem to formula (2.3) implies the following law of finding of two independent events sum probability:

$$P(A + B) = P_A + P_B - P_A \cdot P_B. \tag{2.7}$$

Example. Let there be five black and 10 white balls in one box, and three black and 17 white balls in the other box. The problem is to find the probability of drawing *at least one* black ball upon removing a ball from each box. Using values P_A, P_B and $P(A \cdot B)$, obtained in previous example, we find:

$$P(A + B) = 1/3 + 3/20 - 1/20 = 22/60.$$

§ 2.6. DISCRETE AND CONTINUOUS RANDOM QUANTITIES. DISTRIBUTION SERIES, DISTRIBUTION FUNCTION. PROBABILITY DENSITY

Often, numerical values are connected with outcomes of some experiment. E.g., numbers are on the faces of a cube, so falling of any face is falling of the corresponding number. When you throw the same cube again, the numbers will change randomly. In this case, we speak about a random quantity.

Under a *random quantity* (RQ) we mean a quantity the amount of which depends on results of an experiment with random outcomes.

Random quantities are denoted with capital letters (X, Y...), and their values with lowercase letters (x, y...).

Of the multitude of all random quantities, two most common types are distinguished: *discrete* and *continuous ones.*

***Discrete* random quantity is such RQ that can take only a finite (or countable) set of values.**

These values are numbered x_1, x_2, x_3..., and the probabilities of their appearance are denominated p_1, p_2, p_3...

We will consider discrete values with a *finite* set of values. Examples of such values are the number of letters on a random chosen page of a book, the energy of an electron in an atom, the number of grains in a spike of wheat, etc.

A *continuous* random quantity is such RQ that can take any value in some specific interval (*a*, *b*).

The boundaries of an interval can also take infinitely large values.

Examples of continuous random variables are average air temperature in a certain time interval, mass of grains in a spike of wheat, result of any quantitative analysis in medicine, etc.

Series of discrete random quantity distribution

A discrete random quantity is considered preassigned if all its possible values x_1, x_2...x_N and their corresponding probabilities p_1, p_2...p_N are known. The set of RQ values and their probabilities, specified in the form of a table, is called *distribution series*, or *distribution* of a discrete random quantity:

X	x_1	x_2	x_3	...	x_N
P	p_1	p_2	p_3	...	p_N

The sum of all probabilities is one:

$$\sum_{i=3}^{N} P_i. \tag{2.8}$$

A distribution series is the most complete characteristic of a *discrete* RQ.

Distribution function

The complete characteristic of a continuous random quantity is the *distribution function* $F(x)$, the value of which in each point x is equal to the probability of the random quantity X to take a value less than x:

$$F(x) = P(X < x). \tag{2.9}$$

The probability that the RQ value is less than1 is 0 (the fact of all numbers which are less than $+\infty$ is a certain event), so $F(+\infty) = 1$. The probability of the fact that a value of RQ will be less than $-\infty$, is zero (the fact that there are no such numbers means an impossible event), that is why $F(-\infty) = 0$. A typical form of a distribution function is shown in fig. 2.5.

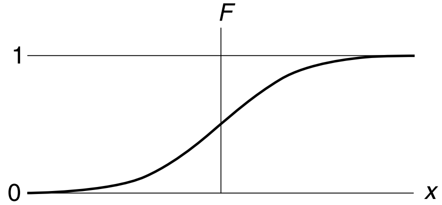

Fig. 2.5. Typical distribution function of a random quantity

The distribution function makes it possible to calculate the probability of a value of a continuous random quantity to fall within the given interval (x_1, x_2):

$$P(x_1 < X < x_2) = F(x_2) - F(x_1). \tag{2.10}$$

Distribution density

Distribution functions of all continuous random quantities are very much alike: they all increase uniformly from 0 to 1. Individual specialties of random quantities are revealed by another function called *distribution density*.

Distribution density **(or probability density)** $\boldsymbol{f(x)}$ **of a continuous random quantity is the derivative of the initial distribution function:**

$$f(x) = \mathrm{d}F/\mathrm{d}x. \tag{2.11}$$

The distribution density has the following probabilistic interpretation:

The probability the continuous random quantity $\boldsymbol{X}$ **to take values within a small interval** $\boldsymbol{(x, x + dx)}$**, is equal to the product of the probability density by the width of the interval:**

$$\mathrm{d}P = f(x) \cdot \mathrm{d}x. \tag{2.12}$$

For plotting a density distribution, the probability of an experiment the value of a continuous random quantity falling within a given interval (x_1, x_2), is equal to the area of the corresponding curvilinear trapezoid (fig. 2.6). For all that, the area under the entire plot curve is equal to *one*. This condition is equivalent to the normalization condition (2.8) for discrete RQs.

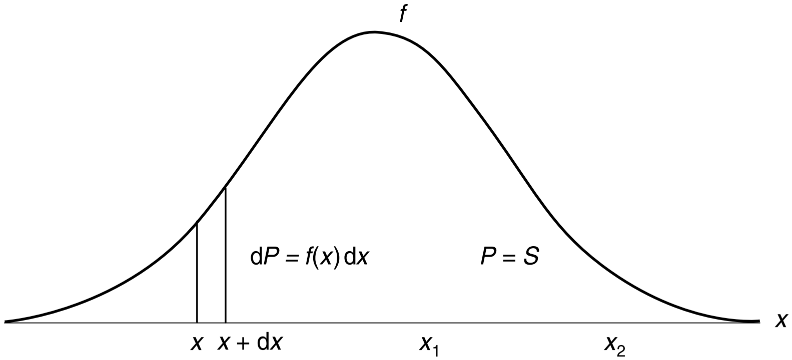

Fig. 2.6. Typical distribution density of a random quantity

For problems of practical statistics, only three types of intervals are of interest: the "left tail" of distribution $(-\infty, x_1)$; "central" interval (x_1, x_2) and "right tail" of distribution $(x_2, +\infty)$.

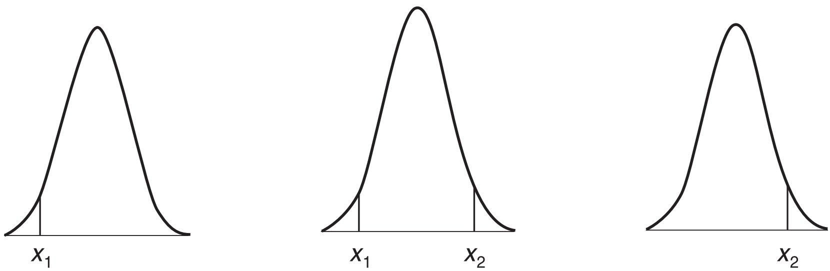

Fig. 2.7. Intervals used in practical statistics

§ 2.7. NUMERICAL CHARACTERISTICS OF RANDOM QUANTITIES

The distribution series and the distribution density contain full information about the corresponding random quantity; nevertheless, in solving many practical problems, it is enough to know two numerical characteristics of the random quantity: *mathematical expectation* and *dispersion*. We will give a not very strict but clear definition of these characteristics.

The mathematical expectation **M_X of a random quantity X is its arithmetic mean.**

This definition has the following meaning. Let a series of n experiments produce n values of a random quantity: x_1, x_2, ... x_n. For unlimited increase in the *length of the series*, the average of all the obtained values tends to M_X:

$$\frac{\sum_i^n x_i}{n} \to M_x \text{ with } n \to \infty. \quad (2.13)$$

Possible values of a random quantity are scattered around its mathematical expectation $M(x)$: one part of them exceeds $M(x)$, the other part is less than $M(x)$. A dispersion of values of a random quantity around its mathematical expectation is estimated by means of a variance.

A variance **is mathematical expectation of the square deviation of a random quantity from its mathematical expectation:**

$$D_x = M[X - M_X]^2. \quad (2.14)$$

The formulas for calculating the variance of discrete and continuous random quantities are as follows:

$$D_x = \sum_{i=1}^{n} p_i \cdot [x_i - M_x]^2, \quad (2.15)$$

$$D_x = \int_{-\infty}^{+\infty} [x_i - M_x]^2 \cdot f(x) \cdot \mathrm{d}x. \quad (2.16)$$

For calculating the variance, the deviations of a random quantity are squared. This is done to suppress the minus sign that appears when $x < M_X$. If this is not done, the negative and positive values will compensate one another and the result will be zero. In order to get rid of the consequences of squaring deviations, after calculating the variance, a square root is extracted from it. The resulting value is used as a measure of deviation of a random quantity from the mean value.

The quadratic deviation **(QD) of a random quantity is the square root of its variance:**

$$\sigma_x = \sqrt{D_x} \quad (2.17)$$

(sometimes the term “standard deviation” is used).

For data processing, mathematical operations are done on random quantities, because of which new random quantities are obtained. Let us show how mathematical expectations and variances change in this case.

1. To add a random quantity with a constant (C), the latter is added to the mathematical expectation and the variance, and QD do not change:

$$M(X+C)=M_X+C;$$
$$D(X+C)=D_X.$$

2. To multiply (divide) a random quantity by a constant (k), the mathematical expectation is multiplied by the constant, and the variance is done by its square:

$$M(k\cdot X)=k\cdot M_X;$$
$$D(k\cdot X)=k^2\cdot D_X,\quad \sigma(kX)=k\cdot\sigma_X.$$

3. To add random quantities (both independent and dependent), their mathematical expectations are added:

$$M(X_1+X_2)=M_1+M_2.$$

4. To add the *independent* random quantities, their variances are added:

$$D(X_1+X_2)=D_1+DX_2.$$

§ 2.8. SOME DISTRIBUTION LAWS OF CONTINUOUS RANDOM QUANTITIES

Let us consider some distribution laws of random quantities, which are important for practical use.

Normal distribution law (Gauss law)

The random quantity X is distributed according to the *normal* law if it is defined on the entire numerical axis and its probability density is determined by the formula:

$$f(x)=\frac{1}{\sigma_x\cdot\sqrt{2\pi}}\cdot\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \tag{2.18}$$

where $\mu = M_X$ is mathematical expectation of the random quantity; σ is its quadratic deviation.

For practical statistics, importance of the normal distribution law is related to the *Central Limit Theorem* according to which the sum of a large number of independent random quantities with the same distribution law has a distribution that can be considered normal. At the same time, the law of distribution to which summands are subject does not matter and can be totally unknown. We will use this property in the next paragraph.

Fig. 2.8 shows plot curves of the probability density of two normally distributed RQ with $\mu = 0$, $\sigma = 2$ and $\mu = 2$, $\sigma = 1$.

Let us note some properties of these plot curves:

- the plot curve of density distribution of the normal law is symmetric and bell-shaped; the line of symmetry passes through the mathematical expectation point of a random quantity ($x = \mu$);
- in the point $x = \mu$ the function attains its maximum value;
- the parameter σ characterizes the shape of the distribution curve: the less σ, the narrower and higher is the plot curve.

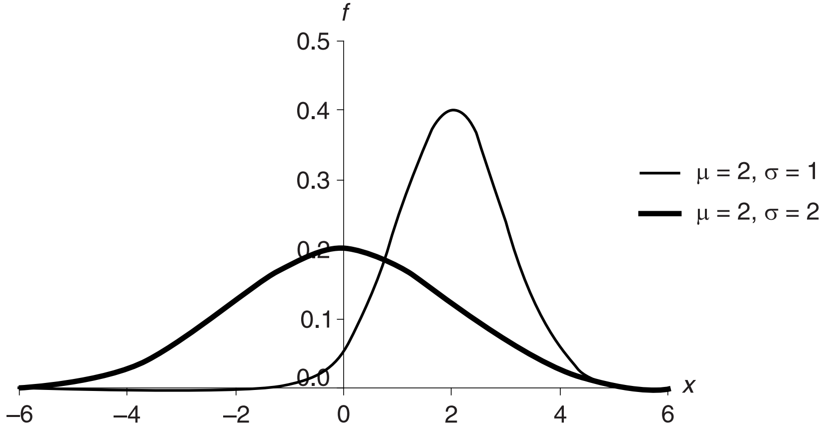

Fig. 2.8. Probability density plot curves for the normal distribution law

Standard computer functions are used to calculate values of the distribution function and the probability density of the normal law values. In the well-known Excel application, these calculations are done by the statistical function NORMDIST (x, μ, σ, m). When $m = 0$, the calculation is done for the *distribution density*, and for $m = 1$ the calculation is done for the *distribution function*.

The normal distribution with $\mu = 0$ and $\sigma = 1$ is called *standard*.

Using the properties of mathematical expectation and variance it is not difficult to show that if random quantity X does not have normal distribution with parameters μ and σ, then random quantity $X_0 = (X - \mu)/\sigma$ has the *standard normal* distribution. Hence, the probability of event $|X - \mu| < k \cdot \sigma$ is equal to the probability of event $|X_0| < k$. Using formula (2.10), we will find

$$P(-k\sigma < |X - \mu| < k\sigma) =$$
$$= \text{NORMDIST}(k, 0, 1, 1) - \text{NORMDIST}(-k, 0, 1, 1).$$

For $k = 1$, $k = 2$ and $k = 3$ we obtain:

$$P(-\sigma < X - a < \sigma) = 0.6826,$$
$$P(-2\sigma < X - a < 2\sigma) = 0.9544, \quad (2.19)$$
$$P(-3\sigma < X - a < 3\sigma) = 0.9974.$$

The last number shows that the probability of deviation of a normally distributed random quantity from the average by more than 3σ amounts to only 0.26%. Relations (2.19) are shown in fig. 2.9.

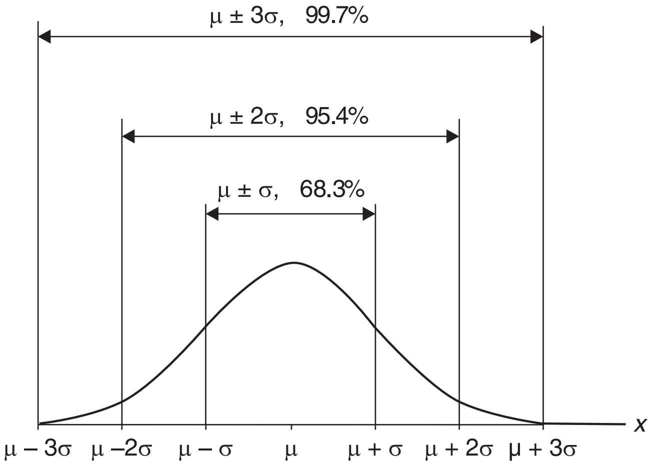

Fig. 2.9. Probabilities of deviation of a normally distributed random quantity from the mathematical expectation

Distribution χ^2, Student distribution and Fisher distribution

The three distributions below are connected with the standard normal distribution, which are of great importance in mathematical statistics.

Distribution χ^2

Let X_1, X_2, ... X_ν be independent random quantities with a *standard normal distribution*. Then the sum of their squares is subject to distribution χ^2 (chi-square):

$$Y = X_1^2 + X_2^2 + \ldots + X_\nu^2. \quad (2.20)$$

The number of summands ν (nu) is called the number of degrees of freedom. Density distribution plot curve for ν = 5 is shown in fig. 2.10.

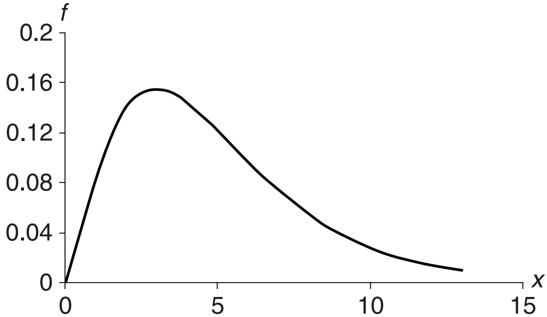

Fig. 2.10. Density distribution χ^2 for $\nu = 5$

Student distribution

If X is a random quantity with standard normal distribution, and Y has distribution χ^2 with the number of degrees of freedom ν, then random quantity

$$Z = \frac{X \cdot \sqrt{\nu}}{\sqrt{Y}} \tag{2.21}$$

is subject to the Student distribution with ν degrees of freedom. A plot curve of the Student density distribution is like a plot curve of standard normal distribution, and is not shown here.

Fisher distribution

If Y_1 and Y_2 are independent random quantities having χ^2-distribution with ν_1 and ν_2 degrees of freedom respectively, then the relation

$$F = \frac{Y_1 \cdot \nu_2}{Y_2 \cdot \nu_1} \tag{2.22}$$

has F-distribution of Fisher. At all that, ν_1 is called the numerator degrees of freedom, and ν_2 is called the denominator degrees of freedom.

Density distribution plot curve of F-distribution for $\nu_1 = 5$ and $\nu_2 = 10$ is presented in fig. 2.11.

Exponential distribution law. Boltzmann distribution

Continuous random quantity with positive values, whose probability density is given by the formula:

$$f(x) = 1 - \lambda \cdot e^{-\lambda x}, \quad x \geq 0, \tag{2.23}$$

is called distributed according to the *exponential* law.

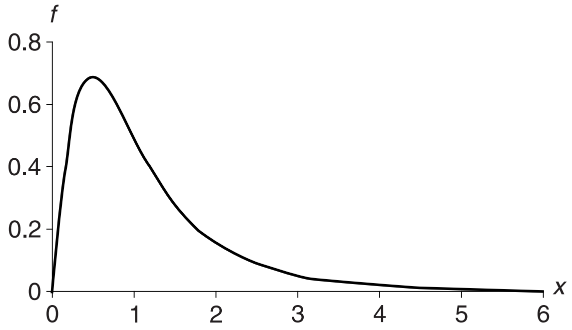

Fig. 2.11. F-distribution density for $\nu_1 = 5$ and $\nu_2 = 10$

The distribution function of the exponential law is expressed by the formula:

$$F(x) = 1 - e^{-\lambda x}, \quad x \geq 0. \tag{2.24}$$

In physics, instead of the distribution function (2.21), the following function is used:

$$F_Б(x) = e^{-\lambda x}, \quad x \geq 0, \tag{2.25}$$

which is equal to the probability that RQ will take a value *higher than x*. By this function, potential energies distribution of particles in the force fields is described. This distribution is called the *Boltzmann distribution*. From the statistical Boltzmann distribution it follows that the barometric formula determing the altitude distribution of gas in the gravitational field of the Earth:

$$n = n_0 \cdot \exp(-mgh/kT), \tag{2.26}$$

where n and n_0 are molecule concentration at altitude h and near the Earth surface; m is the mass of the molecule; k is the Boltzmann constant; T is the absolute temperature.