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Chapter 1

INTRODUCTION TO MATHEMATICAL ANALYSIS

1.1. FUNCTIONS

1.1.1. Definition of function, numerical intervals and neighborhood of points

One of the basic mathematical concepts is the concept of a function, which establishes the relationship between the elements of two sets.

Definition. Let X, Y be some sets, whose elements are some numbers. If each number $x \in X$ is assigned by some law or rule f the corresponding number $y \in Y$, then they establish that, on the set of X, there is a numeric function f, and write this functional dependence with the formula $y = f(x)$ or, more clearly, in the form of the following diagram:

$$X \xrightarrow{f} Y. \qquad (1.1)$$

The variable x is called a *independent variable,* by other words*, an argument*, and the variable y is called *an dependent variable* (on x), by other words, *a function*.

The set of X — a range of the argument variation — is called *the domain of the function* (DOF). The set of Y, containing all the values that y takes, is called *the domain of function change*.

Further, the sets of X and Y are often finite or infinite intervals.

Finite intervals:

- open interval, by other words, simply interval $(a; b)$ is a set of real numbers, satisfying the inequalities of $a < x < b$, by other words, $(a; b) \Leftrightarrow (a < x < b)$, where $\Leftrightarrow$ is the equivalence sign;
- closed interval (by other words, a segment) $[a; b]$: $[a; b] \Leftrightarrow (a \leqslant x \leqslant b)$;
- half-open intervals $(a; b]$ and $[a; b)$: $(a; b] \Leftrightarrow (a < x \leqslant b)$ and $[a; b) \Leftrightarrow (a \leqslant x < b)$ respectively.

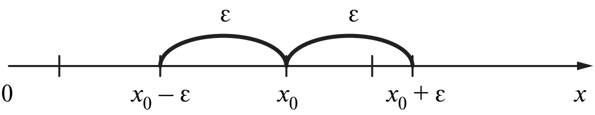

Fig. 1.1. ε-Neighborhood of a point x_0

Infinite intervals:

- $(-\infty, +\infty) = R$ is the set of all real numbers, i.e. $R \Leftrightarrow (-\infty < x < +\infty)$; analogously, $(a; +\infty) \Leftrightarrow (a < x < +\infty)$ etc.

Numbers a, b are called respectively *left and right ends* of these intervals.

Symbols of $-\infty$ and $+\infty$ are not numbers, but express the process of infinite movement of the numeric axis points to the left and to the right from the origin 0.

Let x_0 be any real number (a point on the number axis). *A neighborhood of point* x_0 is any interval $(a; b)$, containing point x_0, interval $(x_0 - \varepsilon; x_0 + \varepsilon)$, where $\varepsilon > 0$, symmetrical about x_0, is called *ε-neighborhood of the point* x_0 (Fig. 1.1).

If $x \in (x_0 - \varepsilon; x_0 + \varepsilon)$, the inequalities are true as follows:

$$x_0 - \varepsilon < x < x_0 + \varepsilon,$$

the latter is equivalent to

$$|x - x_0| < \varepsilon.$$

The particular value of function $f(x)$ when $x = a,$ can be found by substituting a instead of the argument: $f(a)$. For all that, the a can be either an alphanumeric expression or some function, e.g. $\varphi(t)$. In the last case, $f(\varphi(t))$ will be a combined function, which we will meet in Section 1.1.3.

Example 1. Find domain and range of the function

$$y = \sqrt{1 - x^2}.$$

Solution. The domain of this function consists of all x, for which it makes sense. Thus, $X = \{|x| \leqslant 1\} \Leftrightarrow [-1; 1]$, $Y = [0; 1]$, i.e. $[-1; 1] \xrightarrow{f} [0; 1]$.

Example 2. Find the domain and range of the function

$$y_n = \left(\frac{1}{2}\right)^n.$$

Solution. In this case, the independent variable n takes integer positive values $n \in N = \{1, 2, ...\}$, therefore, y is a function of a natural argument and is calculated by the given formula $Y = \left\{\frac{1}{2}, \frac{1}{4}, ..., \left(\frac{1}{2}\right)^n, ...\right\}$, $N \xrightarrow{f} Y$.

Example 3. If $\varepsilon = 0.1$, construct ε-neighborhood of the point $x_0 = 2$.

Solution. Under the definition, ε-neighborhood of the point $x_0 = 2$ will be interval $|x - 2| < 0.1$, i.e. $-0.1 < x - 2 < 0.1 \Rightarrow$ $\Rightarrow 1.9 < x < 2.1$ (Fig. 1.2).

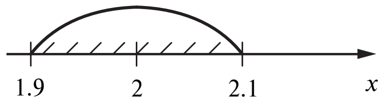

Fig. 1.2. Interval $|x - 2| < 0.1$

Self-study work

1. Construct intervals of change for the variable x, satisfying the inequalities:

1) $|x| < 4$; 2) $x^2 \leqslant 9$;

3) $|x - 4| < 1$; 4) $-1 < x - 3 \leqslant 2$;

5) $x^2 > 9$; 6) $(x - 2)^2 \leqslant 4$.

2. Find the domain of function:

1) $y = \sqrt{x+2}$; 2) $y = \sqrt{9 - x^2}$;

3) $y = \sqrt{4x - x^2}$; 4) $y = \sqrt{-x} + \sqrt{4 + x}$;

5) $y = \arcsin\frac{x-1}{2}$; 6) $y = -\sqrt{2\sin x}$;

7) $y = -\frac{x\sqrt{16 - x^2}}{2}$; 8) $y = \sqrt{x+1} - \sqrt{3 - x}$.

3. Calculate function values at the points given:

1) $f(x) = x^2 - x + 1$; $f(2)$, $f(a+1)$;

2) $\varphi(x) = \frac{2x - 3}{x^2 + 1}$; $\varphi\left(\frac{3}{2}\right)$, $\varphi\left(\frac{1}{x}\right)$, $\frac{1}{\varphi(x)}$;

3) $F(x) = x^2$; $\frac{F(b) - F(a)}{b - a}$, $F\left(\frac{a+h}{2}\right) - F\left(\frac{a-h}{2}\right)$.

1.1.2. Some properties of functions and their graphs

Let function $f : X \to Y$ be given. The rule for finding y, knowing x, can be defined by the function graph.

Definition. *The function graph* in a rectangular Cartesian coordinate system is a set of all points, whose abscissas are the values of the argument, and ordinates are the corresponding values of the function.

Example 1. The $y = x^2$ function graph is the parabola, which axis of symmetry coincides with the positive semi-axis of ordinates, and the vertex does with the origin of the coordinates system (Fig. 1.3).

Often, graphs are automatically drawn by self-writing devices or displayed on a monitor screen. The advantage of this method is clearness, the disadvantage is inexactness.

A function can also be defined using tables or formulas (analytically). *The tabular method* is used practically when processing results of observations of the approximate function values. *The analytical method* is most convenient for the complete curve sketching using mathematical analysis methods.

Let us introduce the basic characteristics of a function: monotonicity, boundedness, a property of being even (odd), periodicity.

Definition. A function is called *increasing* (*decreasing*) on an interval, if the larger value of the argument from this interval corresponds to the larger (smaller) value of the function.

The graph of a function increasing on an interval $(a; b)$, if it is viewed from the left to right rises (Fig. 1.4, *a*), and for a decreasing function the graph goes down (Fig. 1.4, *b*).

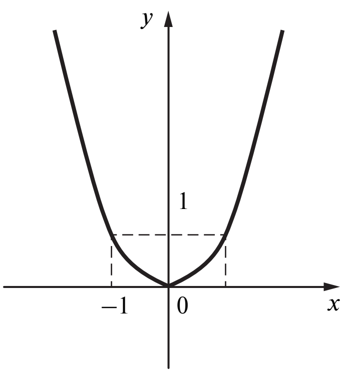

Fig. 1.3. Graph of the function $y = x^2$

Definition. The interval of the independent variable, which function increases (decreases) on, is called *the interval of increase (decrease)*. Both intervals of increase and decrease are called *monotonicity intervals* of a function, and the function on these intervals is called a *monotonic function*.

Definition. The value of an argument at which function becomes zero, is called *zero of a function*.

If function is defined by the formula $y = f(x)$, then zero (or zeros) of the function can be found by solving the equation $f(x) = 0$.

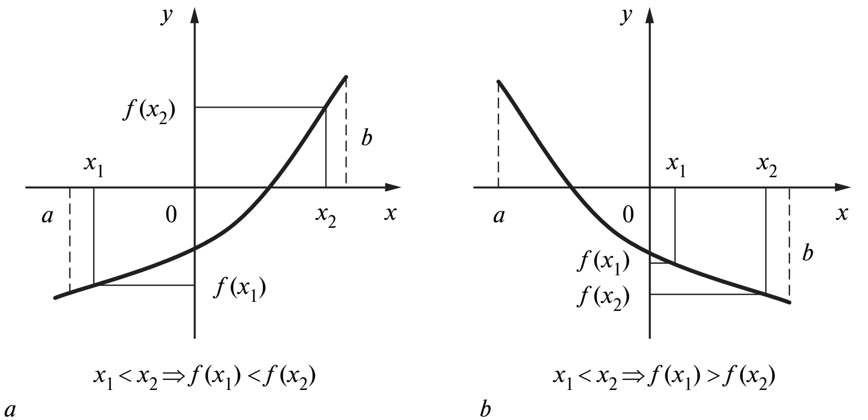

Fig. 1.4. Graphs: *a* is a function, increasing on an interval (a; b); *b* is a function, decreasing on an interval (a; b)

When the definition is graphical, zeros of the function are the points where the graph is crossed by the x-axis.

Example 2. Find zeroes of function $y = 2x + 1$.
Solution.

$$2x + 1 = 0 \Rightarrow x = -\frac{1}{2} \text{ (Fig. 1.5).}$$

Definition. A function is called *even* if the value of a function does not change for changing the sign of an allowed value of the argument. A function is called *odd* if changing the sign of the allowed value of the argument changes the sign of the function value.

So, if the function $f(x)$ is even, then for all x of its domain, the equality $f(-x) = f(x)$ should be true, as it happens, e.g., when $f(x) = x^2$, and if $f(x)$ is odd, then $f(-x) = -f(x)$ is true for any x of the domain of the function, like, e.g., in the case of $f(x) = x^3$.

Notice, that both even and odd functions are without fail defined in the domain, which is symmetrical about the origin of the coordinates system.

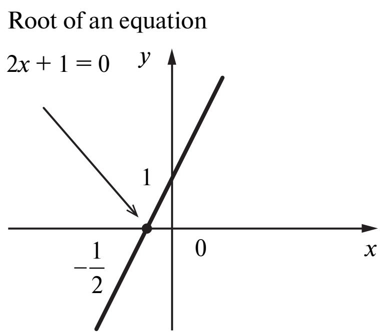

Fig. 1.5. Graph of the function $y = 2x + 1$

At the same time, the graph of an even function is symmetrical about the ordinate axis (similarly in Fig. 1.3), and the function graph of an odd function is symmetrical about the origin of the coordinates system (as in Fig. 1.6).

Notice, that not all functions can be even or odd. Such functions (neither odd, nor even ones) we will call *general functions*.

Definition. Function $f(x)$, defined on the set of X, is called *bounded* on this set, if there is such a number $M > 0$, that inequality $|f(x)| \leqslant M$ is true for all $x \in X$.

A graph of the bounded function is located between straight lines $y = M$ and $y = -M$ (Fig. 1.7).

Definition. Function $f(x)$ is called *periodic*, if there is such a positive number a, that $f(x + a) = f(x) = f(x - a)$ for any x from the DOF (points x, $x + a$, $x - a$ belonging to the domain of a function). At the same time, the smallest positive number a with such a property (if any) is called *a function period*.

A periodic function graph is obtained by repeating the part of the graph, corresponding to the interval of the abscissa axis, which is equal in length to the function period.

An example of a periodic function is the function y = cos x, defined on the real axis, whose period is 2π (Fig. 1.8).

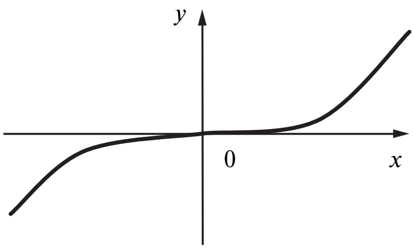

Fig. 1.6. Graph of the function $y = x^3$

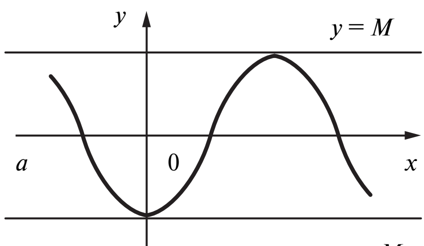

Fig. 1.7. Graph of the function of the bounded function

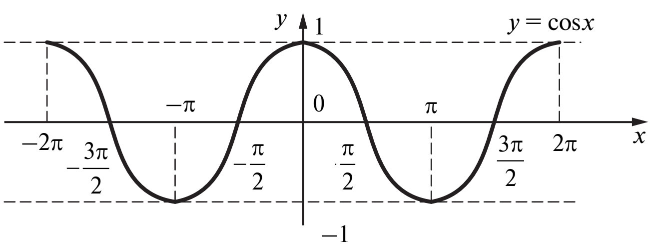

Fig. 1.8. Graph of the function $y = \cos x$

Thus, a shift of the graph of a periodic function along the abscissa axis by an interval, whose length is a multiple of a period, does not change this graph. In particular, the domain of a periodic function is not bounded.

Self-study work

1. Indicate, which of the following functions are even or odd:

1) $\dfrac{\sin x}{x}$; 2) $\dfrac{a^x - 1}{a^x + 1}$;

3) $a^x + \dfrac{1}{a^x}$; 4) $a^x - \dfrac{1}{a^x}$;

5) $x \sin^2 x - x^3$; 6) $x + x^2$;

7) $|x|$; 8) $\dfrac{\tan x}{\sin 2x}$.

2. Find the function zeroes:

1) $y = ax + b$; 2) $y = x^2 + px + q$;
3) $y = x^4 + px^2 + q$; 4) $y = 2 \log_{10}(x + 1)$;
5) $y = a^{2x} - a^2$ $(a > 0)$; 6) $y = 2\sin x - 1$;
7) $y = \tan x + 1$.

3. Find function period:

1) $y = \tan 2x$; 2) $y = \sin\dfrac{x}{2}$; 3) $y = \dfrac{\cot x}{\cos 2x}$.

4. Using properties of the odd and even function graphs and the results of Section 1.1.2, construct the following function graphs:

1) $y = |x|$; 2) $y = -x + |x|$;
3) $y = -|x - 2|$; 4) $y = x - 4 + |x - 2|$, $x \in [-2; 5]$;
5) $y = \log_{10}(x + 2)$; 6) $y = 2^{-x}$;
7) $y = x^2 + 2x + 2$; 8) $y = -x^2 + 4x$.

1.1.3. Composite function. Inverse function

Definition. *Composite function* (Fig. 1.9) is a function, whose argument is also a function, i.e. $F(x) = f(\varphi(x))$, by other words, in the form of a diagram, similarly to the formula (1.1).

In other words, in order to calculate the value at point x of the composite function $f(\varphi(x))$, consisted of functions f and φ, we should firstly find the particular

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[rr, bend left, "F"] & U \arrow[r, "f"] & Y \end{tikzcd}$$

Fig. 1.9. Composite function

value $u = \varphi(x)$ of inner function φ, and then substitute it as an argument in the outer function f.

In this case, the domain of function $F(x)$ should be selected so that the intermediate set U, on the one hand, is the range of function $\varphi(x)$, and, on the other hand, is the domain of function $f(u)$.

Example 1. Consider composite function $y = \log_{10}(1 - x^2)$. In this case, $y = f(u) = \log_{10} u$, while $u = \varphi(x) = 1 - x^2$. The domain of the function is y is interval $(-1, 1)$, in which both function $\varphi(x)$ and function $f(u)\big|_{u-\varphi(x)}$ make sense.

Let us consider a function with domain X and range Y. Let us assume, that each value $y \in Y$ corresponds to one definite point $x \in X$, as such $y = f(x)$. Then, there exists a function φ: $Y \to X$, which translates any $y \in Y$ to $x \in X$, that meets the property $y = f(x)$ mentioned above.

Functions f and φ with the properties above-cited are called *reciprocal*, and function φ is called *the inverse* of f. Having given the fact, that symbol x, as a rule, corresponds to an independent variable, the format $y = \varphi(x)$ is as a rule used instead of $x = \varphi(y)$.

From the definition of inverse function it follows, that any strictly monotonic function has an inverse one.

There is a simple relationship between function $y = f(x)$ and $y = \varphi(x)$ graphs: the graph of inverse function $y = \varphi(x)$ is symmetrical to the graph of function $y = f(x)$ given about the bisector of the angles formed by quadrants I and III.

Notice, that reciprocal functions f and φ meet the relation given below and can be calculated as follows:

$$f(\varphi(x)) = \varphi(f(x)) = x. \tag{1.2}$$

Example 2. Let $y = f(x) = x^3$. Then $f(\varphi(x)) = \varphi^3(x)$ and the equality (1.2) gives $\varphi^3(x) = x$, or $\varphi(x) = x^{1/3}$, which, however, follows easily from the relation $y = x^3$ (Fig. 1.10) directly.

It is important to keep in mind that function $f(x)$, increasing or decreasing on set X, has an inverse function known to be (the definition of increasing and decreasing functions is given in Section 1.1.2).

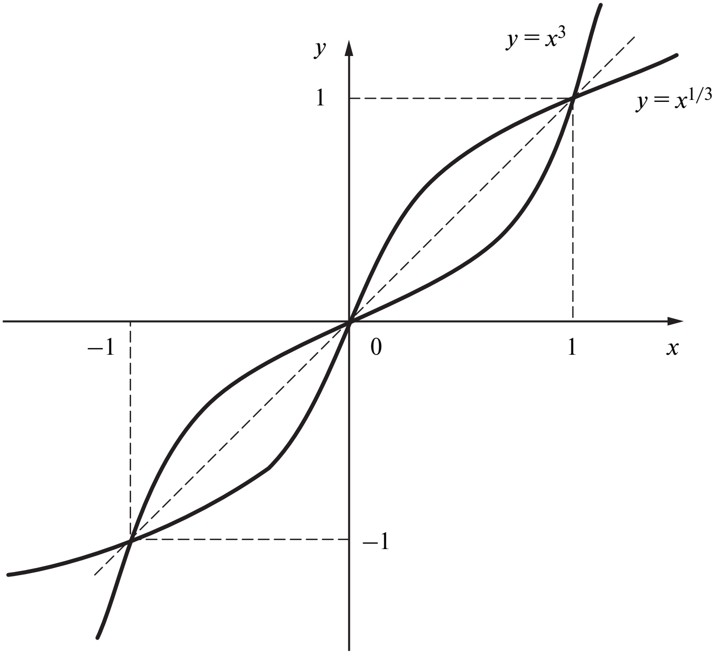

Fig. 1.10. Graphs of reciprocal functions $y = x^3$ and $y = x^{1/3}$

Otherwise, the uniqueness of the correspondence between X and Y is violated, and the inverse function does not exist. However, as a rule, domain X can be divided into intervals, where function $f(x)$ increase or decrease, and on each of them the inverse function can be defined.

Example 3. Let $y = x^2$. Then $X = (-\infty;\ \infty)$, and $Y = [0;\ +\infty)$. Thus, no one-to-one correspondence between X and Y (each $y \neq 0$ is associated with two x values, differing in signs), and, therefore, no inverse function (Fig. 1.11) too.

If X is divided into $(-\infty;\ 0]$ and $[0;\ +\infty)$, then on each half-line, there is one-to-one dependence $y = x^2$. Therefore, on ray $(-\infty;\ 0]$, function $y = x^2$ has inverse function $y = -\sqrt{x}$, and on ray $[0;\ +\infty)$ the inverse function of it is one $y = \sqrt{x}$.

Example 4. Let function y with independent variables x is expressed by linear dependence $3x + 2y - 6 = 0$. Find the inverse function and construct graphs of direct and inverse functions.

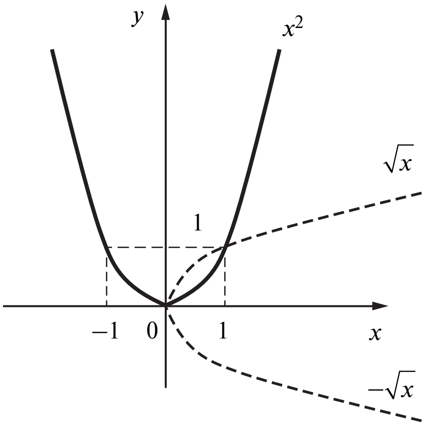

Fig. 1.11. Example of a function, which does not have a reciprocal function

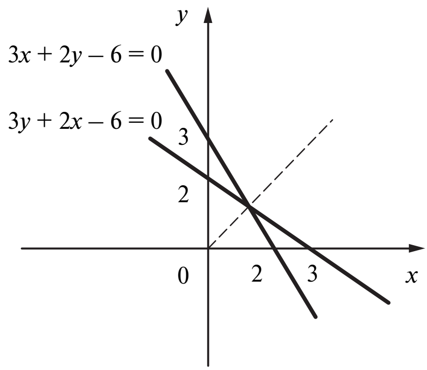

Fig. 1.12. Graphs of the reciprocal linear functions

Solution. To find the inverse function in a common coordinate system with the direct function, it is sufficient to exchange x and y in the corresponding equation.

Thus, in our example, the inverse relationship is expressed by relation $3y + 2x - 6 = 0$ and is also linear.

Constructing the graphs (Fig. 1.12), it was taken into account that a straight line is uniquely defined by any pair of different points lying on it. In particular, the straight line $3x + 2y - 6 = 0$ is defined by points (0; 3) and (2; 0).

Note, that in accordance with properties of reciprocal functions, the straight lines in Fig. 1.12 are symmetrical about the bisector of the angles formed by quadrants I and III.

1.1.4. Elementary functions

Definition. The basic elementary functions are the following:

1) power function: $y = x^n$, when n is a real number, $x > 0$ (in some cases, particularly, with natural n, the power function is defined on the whole real axis);
2) exponential function: $y = a^x$, where $a > 0$, $a \neq 1$, and $X = R$;
3) logarithmic function: $y = \log_a x$, when a logarithm base is $a > 0$, $a \neq 1$, and $X = (0; +\infty)$;
4) trigonometric functions $y = \sin x$, $y = \cos x$, $y = \tan x$ and $y = \text{cotan}\, x$;

5) inverse trigonometric functions: $y = \arcsin x$, $y = \arccos x$, $y = \arctan x$ and $y = \operatorname{arccotan} x$.

The set of *elementary functions* includes all the basic elementary functions and constants, as well as all the functions derived from them using four arithmetic operations and the operation of taking a function from a function applied sequentially a finite number of times.

E.g., the functions $y = \log_{10}\left(x + \sqrt{1+x^2}\right)$, $y = \arctan\dfrac{2x^4}{1-x^2}$ and $y = \tan x - \sqrt{x}$ are elementary. The function

$$y = \begin{cases} x, & x \le 1; \\ x^2, & x > 1 \end{cases}$$

is not elementary.

The domain of an elementary function is all the values of the argument for which this function makes sense.

E.g., the domain of function $y = \sqrt{x^2 - 1}$ is set $X = (-\infty < x \leqslant -1 \cup 1 \leqslant x < +\infty)$. In this case, the symbol $\cup$ means the union of the intervals.

Let us consider power and exponential functions.

Power function $y = x^n$ with integer n is defined on the whole real axis; it is even, if n is even, and it is odd, if n is odd (see Fig. 1.2 and 1.5).

When n is arbitrary, the function is considered in the area $x > 0$. If $n > 0$, then the function graphs $y = x^n$ increase from zero to infinity on the interval $(0; +\infty)$, pass through points $(0; 0)$ and $(1; 1)$ and are divided by straight line $y = x$ into curves, that are convex downward when $n > 1$ and convex upward if $0 < n < 1$ (Fig. 1.13, *a*).

If $n < 0$, then function $y = x^n = \left(\dfrac{1}{x}\right)^{|n|}$ decreases from infinity to zero (Fig. 1.13, *b*).

The inverse function to function $y = x^n$, $x > 0$, is the one $y = x^{\frac{1}{n}}$.

We are reminding, that

$$\left(x^a\right)^b = x^{ab}; \tag{1.3}$$

$$x^{\frac{n}{m}} = \sqrt[m]{x^n} \tag{1.4}$$

(n and m are positive integers) and, particularly, $\sqrt[n]{x} = x^{\frac{1}{n}}$.

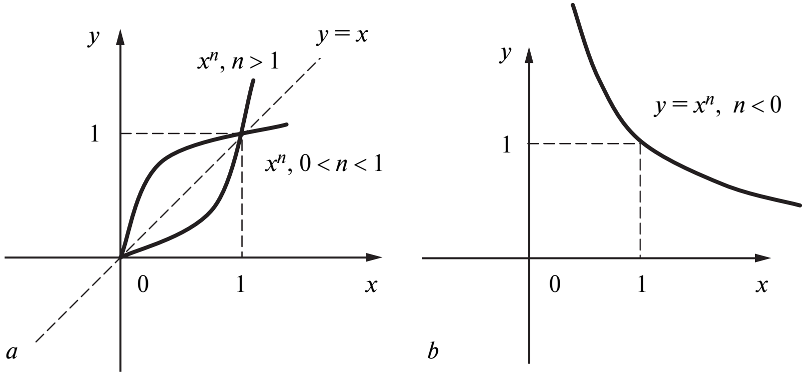

Fig. 1.13. Graph of the function $y = x^n$: a is for $n > 0$; b is for $n < 0$

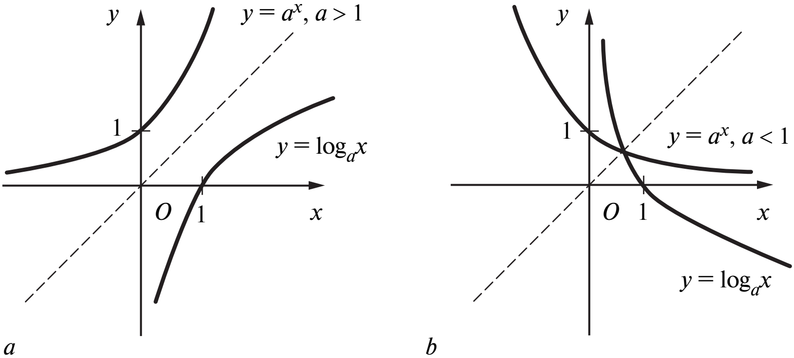

Fig. 1.14. Exponential and logarithmic functions graphs: a is for $a > 0$; b is for $a < 0$

Exponential function $y = a^x$, $-\infty < x < \infty$, and logarithmic one $y = \log_a x$, $x > 0$, when parameter a is the same, are reciprocal. Their graphs are symmetrical about the bisector of the angles formed by the quadrants I and III (Fig. 1.14).

An exponential function is always positive, so its graph is located above the axis Ox. In addition, since $a = 1$, it passes through the point (0; 1). For $a > 1$, an exponential function increases from zero to infinity, and for $a < 1$ it decreases from infinity to zero. Notice, that graph of an exponential function with base a is symmetrical about axis Oy to the exponential function graph with base $\frac{1}{a}$, which follows from the equality $\left(\frac{1}{a}\right)^x = a^{-x}$ (Fig. 1.15).

We are reminding, that function $y = e^x$ ($e = 2.718...$) is called *exponential*, and its graph is called *an exponential curve*; logarithms with base e are find as $\log x$ and called *natural one*. Logarithms with base 10 are designated by $\log_{10} x$ and called as *decimal* one. Thus, $\log_e x = \log x$, $\log_{10} x = \log x$.

Taking into account that logarithmic and exponential functions are reciprocal, we have (see (1.2))

$$\log_a a^x = x, \quad a^{\log_a x} = x, \qquad (1.5)$$

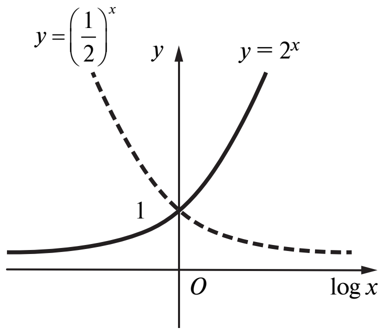

Fig. 1.15. Graph of the function $y = 2^x$ and $y = 2^{-x}$

where the first equality is true for any x and the other one for $x > 0$.

In particular, $x = e^{\log x}$ and, therefore (see (1.3)),

$$x^n = e^{n \log x}, x > 0 \qquad (1.6)$$

(it is the representation of power function with exponential one).

The following formula is true

$$\log_a x = \frac{\log_b x}{\log_b a}, \qquad (1.7)$$

i.e., logarithms of numbers with different bases (a and b, respectively) are proportional to each other with a proportionality coefficient (*transition module*) $\frac{1}{\log_b a} = \log_a b$.

Example 1. Express $\log_2 x$ through $\log_{10} x$ and $\log x$.

Solution.

$$\log_2 x = \frac{\log_{10} x}{\log_{10} 2} = \log_2 10 \cdot \log_{10} x,$$

$$\log_2 x = \frac{\log x}{\log 2} = \log_2 e \cdot \log x.$$

Example 2. Write function $y = 2^x$ as an exponential function with base 10.

Solution. $2^x = 10^{\log_{10} 2^x} = 10^{\log_{10} 2 \cdot x}$.

1.1.5. Trigonometric functions

First, let us remind that, in mathematical analysis, the argument of trigonometric functions is always taken a radian arc or angle measure, i.e. a number equal to the ratio of the length of this arc to the radius of a circle. In this way,

$$\alpha_{\text{rad}} = \alpha^\circ \frac{\pi}{180}, \text{ or } \alpha^\circ = \frac{180}{\pi}\alpha_{\text{rad}}, \tag{1.8}$$

where α° is a degree measure, and α_{rad} is a radian measure of the angle. Particularly,

$$\frac{\pi}{4} \Leftrightarrow 45^\circ,\ \frac{\pi}{3} \Leftrightarrow 60^\circ,\ \frac{\pi}{6} \Leftrightarrow 30^\circ,\ \frac{\pi}{2} \Leftrightarrow 90^\circ \text{ etc.}$$

Trigonometric functions are periodic: $\sin x$ and $\cos x$ have the period of 2π (Fig. 1.16), at the same time, $\tan x$ and $\text{cotan}\, x$ have the period of π (Fig. 1.17).

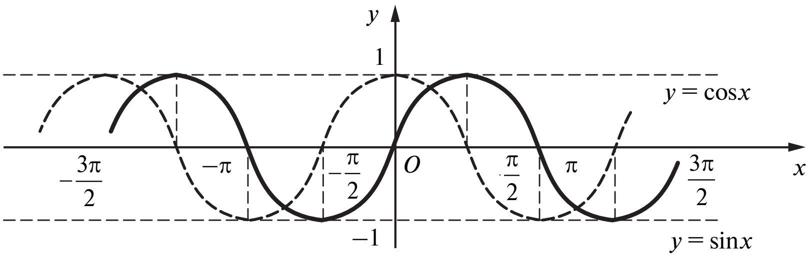

Fig. 1.16. Graph of the function $y = \sin x$ and $y = \cos x$

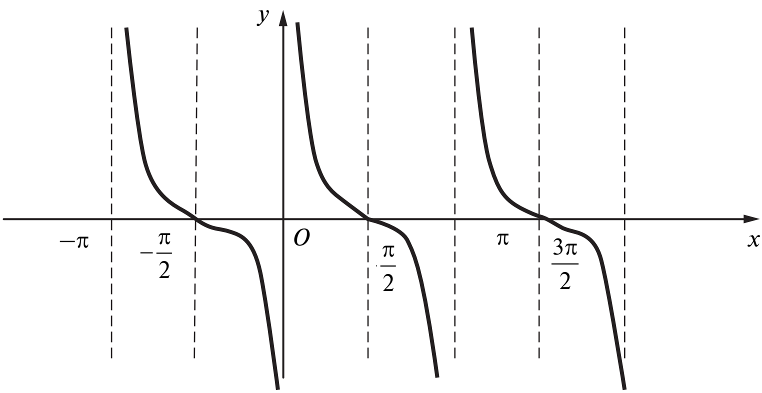

Fig. 1.17. Graph of the function $y = \text{cotan}\, x$

A cosine graph differs from the sine graph by the shift to the left along the axis Ox by $\frac{\pi}{2}$, because $\cos x = \sin\left(x+\frac{\pi}{2}\right)$ (see Section 1.1.2).

Functions sin x, tan x and cotan x are odd, though function cos x is even. Finally, sin x and cos x are defined for any x, tan x — for all x, except for the points such as $(2k+1)\frac{\pi}{2}$, where k is any integer, and cotan x is defined for any x, except the points such as the kind $k\pi$.

We are reminding, that

$$\cos x = \sin\left(\frac{\pi}{2} \pm x\right); \quad \sin x = \cos\left(\frac{\pi}{2} - x\right);$$

$$\begin{gathered} \sin^2 x + \cos^2 x = 1; \quad \sin 2x = 2\sin x \cos x; \\ \cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1; \end{gathered} \tag{1.9}$$

$$\tan x = \frac{\sin x}{\cos x}; \quad \tan 2x = \frac{2\tan x}{1-\tan^2 x};$$

$$\cot x = \frac{\cos x}{\sin x}; \quad \cot x = \tan\left(\frac{\pi}{2} - x\right).$$

1.1.6. Inverse trigonometric functions

Since trigonometric functions are periodic, each value of the function corresponds to an infinite number of argument values. Thus, no one-to-one correspondence between x and y, and, therefore, a single-valued inverse function cannot be defined.

To solve the problem of finding x by y, if $y = \sin x$, they do the following.

Function $y = \sin x$ on the interval $-\pi/2 \leqslant x \leqslant \pi/2$ increase (Fig. 1.18) and, therefore, has an inverse function, find by $y = \arcsin x$.

Taking in account that the graph of the inverse function is symmetrical to the graph of a direct function about the bisector of the angle formed by quadrants I and II , we can find the graph of function arcsin x.

Domain X of the function $y = \arcsin x$ is segment $[-1;\ 1]$, and the range of Y is segment $[-\pi/2;\ \pi/2]$, i.e.,

$$[-1;\ 1] \xrightarrow{\arcsin x} [-\pi/2;\ \pi/2].$$

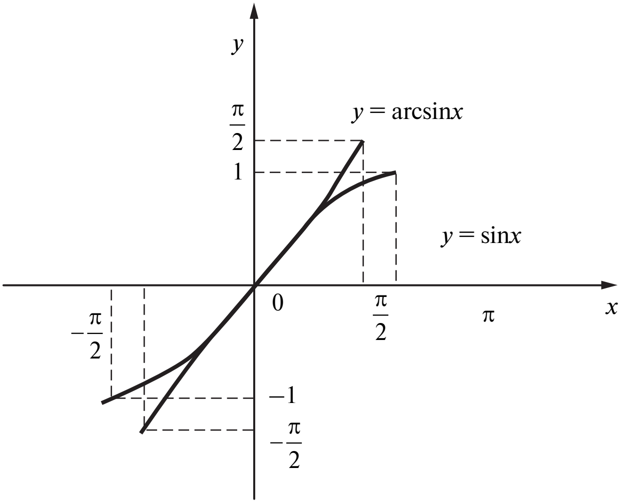

Fig. 1.18. Graph of the function $y = \arcsin x$

In addition, $y = \arcsin x$ is an odd and increasing function.

The value of the function $\arcsin x$ is the radian measure of the angle, the sine of which is equal to the value given of the independent variable x; at the same time, among all the angles satisfying this condition, only the angle of the segment $[-\pi/2; \pi/2]$ is chosen, i.e.

$$y = \arcsin x \Leftrightarrow \sin y = x; |y| \leqslant \pi/2. \quad (1.10)$$

All values of y, which satisfy the equation $\sin y = x$, are found by the formula

$$y = \pi k + (-1)^k \arcsin x; k = 0, \pm 1, \pm 2, ... \quad (1.11)$$

E.g., the solution of the equation $\sin y = \frac{1}{2}$, $|y| \leqslant \pi/2$, is $\arcsin\frac{1}{2}$, i.e. number $\pi/6$. The general solution of the equation $\sin y = \frac{1}{2}$ will be numbers $y = \pi k + (-1)^k \frac{\pi}{6}$, or ..., $-1\frac{5}{6}\pi$, $-1\frac{1}{6}\pi$, $\frac{\pi}{6}$, $\frac{5}{6}\pi$, $2\frac{1}{6}\pi$, ...

The function, which is inverse of $y = \cos x$, is defined similarly. Then we have

$$\left[-1; 1\right] \xrightarrow{\arccos x} \left[0; \pi\right]$$

or

$$y = \arccos x \Leftrightarrow \cos y = x;\ 0 \leqslant y \leqslant \pi.$$

Function $y = \arccos x$ (Fig. 1.19) and satisfies the equality of

$$\arccos(-x) = \pi - \arccos x. \qquad (1.12)$$

The general solution of equation $\cos y = x$, $|x| \leqslant 1$, has the form

$$y = 2\pi k \pm \arccos x. \qquad (1.13)$$

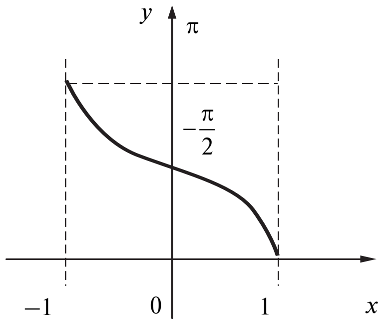

Fig. 1.19. Graph of the function $y = \arccos x$

Notice the formulas:

$$\sin(\arccos x) = \cos(\arcsin x) = \sqrt{1-x^2};$$
$$\arcsin x + \arccos x = \frac{\pi}{2}. \qquad (1.14)$$

Function $y = \arctan x$ is defined on the whole real axis; the range of the function is the open interval $(-\pi/2;\ \pi/2)$, i.e.

$$R \xrightarrow{\arctan x} [-\pi/2;\ \pi/2].$$

This function is increasing and odd (Fig. 1.20).

The numbers of the form

$$y = \pi k + \arctan x. \qquad (1.15)$$

satisfy the equation $\tan y = x$.

Analogously, function $y = \arctan x$ is defined on the whole real axis and takes values on interval $(0;\ \pi)$. It decreases (Fig. 1.21) and satisfied to equality

$$\operatorname{arccot}(-x) = \pi - \operatorname{arccot} x. \qquad (1.16)$$

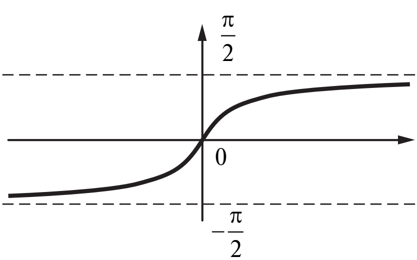

Fig. 1.20. Graph of the function $y = \arctan x$

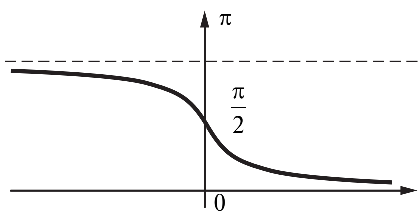

Рис. 1.21. Graph of the function $y = \operatorname{arccotan} x$

The general solution of equation cotan $y = x$ has the form of

$$y = \pi k + \operatorname{arccot} x. \tag{1.17}$$

Notice the formulas:

$$\tan(\operatorname{arccot} x) = \cot(\arctan x) = \frac{1}{x};$$

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2}; \tag{1.18}$$

$$\arctan x = \operatorname{arccot}\frac{1}{x}; \quad \operatorname{arccot} x = \arctan\frac{1}{x}.$$

1.2. LIMITS

1.2.1. Limit of function

The theory of limits is of fundamental importance in mathematical analysis. With its help, such properties of a function as continuity, differentiability, integrability etc. are determined.

Let us consider an example. Let the function be given by $f(x) = \dfrac{x^2 - 1}{x - 1}$, defined for any x, except of $x = 1$. Let's examine the behavior of this function x, when values x that do not differ from 1 much. To do this, we will make a table of function values on the interval that we are interesting in (Table 1.1).

We see that the closer x approaches 1, the closer $f(x)$ values are to 2. In such cases, the number 2 is said to be the limit of function $f(x)$ for x, tending to 1 (or more briefly: $f(x) \to 2$ for $x \to 1$).

We can now give a strict definition of a function limit.

Definition. Let function $f(x)$ is defined in some neighborhood of point x_0, except, maybe, point x_0 itself. The number b is called the *function limit* in point x_0 (or when $x \to x_0$), if for any positive ε, however small it is, the inequality $|f(x) - b| < \varepsilon$ is true for all $x \neq x_0$ of a certain neighborhood of the point x_0.

Table 1.1

x	0.97	0.98	0.99	1.01	1.02
$f(x)$	1.97	1.98	1.99	2.01	2.02

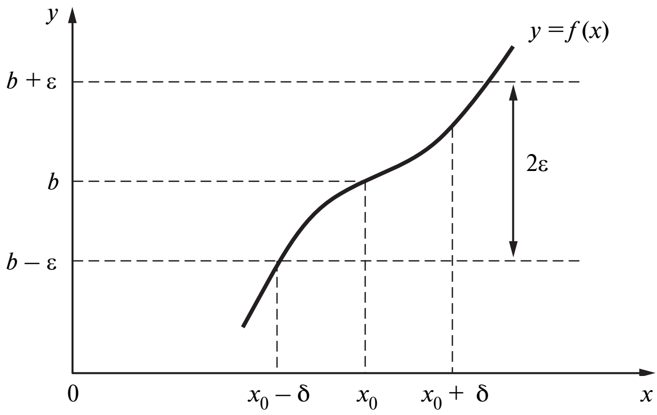

Fig. 1.22. Geometrical meaning of the function limit at point x_0

It is written as follows: $\lim\limits_{x \to x_0} f(x) = b.$

Geometrical meaning of this definition: for any ε-neighborhood of point b (Fig. 1.22) there exists a certain neighborhood of point x_0 (e.g., δ-neighborhood), such that for all $x \neq x_0$ of that neighborhood the corresponding points of the graph of $f(x)$ locates inside a range of 2ε wide, limited by straight lines $y = b + \varepsilon$, $y = b - \varepsilon$.

This definition does not specify how x approaches x_0: from the left, from the right, or oscillation about x_0. But sometimes it is essential.

Definition. Number b_1 is called *function $y = f(x)$ limit from the left at point* x_0, if $f(x) \to b_1$, when $x \to x_0$, being less than x_0.

It is written as follows: $\lim\limits_{x \to x_0 - 0} f(x) = b_1.$

A function limit from the right is defined and written similarly: $\lim\limits_{x \to x_0 + 0} f(x) = b_2$, if $f(x) \to b_2$, when $x \to x_0$, being more than x_0.

Left and right function limits are called *one-sided* (Fig. 1.23). Obviously, if there exists $\lim\limits_{x \to x_0} f(x) = b$, then both one-sided limits also exist and are equal to b.

The converse is also true: if there exists one-sided limits, both equal to b, then $\lim\limits_{x \to x_0} f(x) = b.$

If one-sided limits are not equal to each other ($b_1 \neq b_2$), then $\lim\limits_{x \to x_0} f(x)$ does not exist.

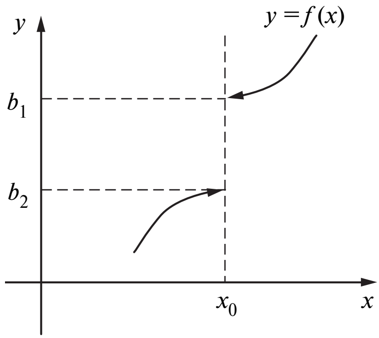

Fig. 1.23. Illustration to the definition of one-sided limits

If the function $y = f(x)$ is defined on interval $(a; +\infty)$, then the function limit can be defined for $x \to +\infty$.

Definition. Number b is called *function limit when* $x \to +\infty$ (i.e. $\lim\limits_{x\to+\infty} f(x) = b$), if for whatever small $\varepsilon > 0$ and any sufficiently large x inequality $|f(x) - b| < \varepsilon$ is true.

In Fig. 1.24, function values $f(x)$ for all $x > E$ are inside the ε-neighborhood of point b.

The function limit for $x \to -\infty$ is defined similarly.

If function $f(x)$ limits for $x \to +\infty$ and $x \to -\infty$ exist and are equal, e.g., to A, then they say, that $f(x)$ has limit A or $x \to \infty$, and they write simply $\lim\limits_{x\to\infty} f(x) = A$.

In definitions of a function limits given before, the limits are assumed to be finite. If function $f(x)$ increases or decreases infinitely when $x \to x_0$, the limit of $f(x)$ is said to be equal to infinity ($\lim\limits_{x\to x_0} f(x) = \infty$).

Among functions, that have limits (at a certain point or ∞), a class of functions, which have a limit equal to 0, is selected. Such functions are called *infinitely small* ones (infinitesimals) and are find by letters α, β, γ etc.

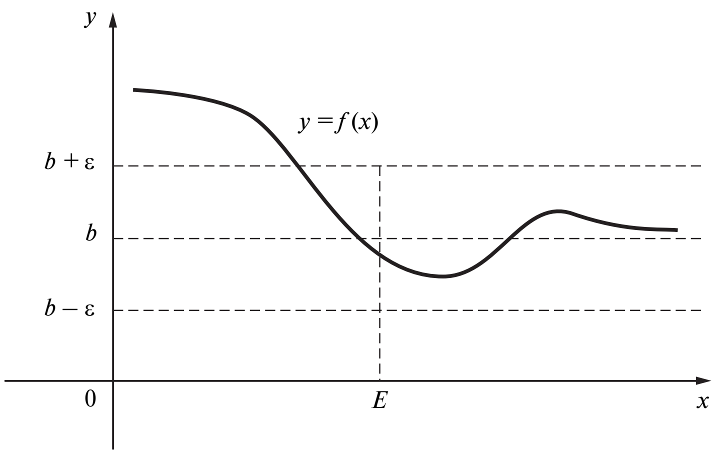

Fig. 1.24. Function limit when $x \to +\infty$

The concept of equivalence of infinitesimals is often used in calculating limits.

Definition. Let $\alpha(x)$ and $\beta(x)$ be infinitely small functions. If $\lim\limits_{x\to x_0}\frac{\alpha(x)}{\beta(x)}=1$, then $\alpha(x)$ and $\beta(x)$ are called *equivalent* ones (when $x \to x_0$).

1.2.2. Basic theorems on limits

This section is devoted to the basic properties of function limits. Such rules give us a possibility to calculate limits of functions defined by algebraic operations on a variable. In the theorem, which will be given further, functions $f(x)$, $g(x)$ are assumed to have a common domain containing the point x_0, and have finite limits at this point.

Theorem 1.1. A limit of the sum of two functions is equal to the sum of their limits.

$$\lim_{x\to x_0}\big(f(x)+g(x)\big)=\lim_{x\to x_0}f(x)+\lim_{x\to x_0}g(x).$$

Theorem 1.2. The limit of the product of two functions is equal to the product of their limits.

$$\lim_{x\to x_0}\big(f(x)\cdot g(x)\big)=\lim_{x\to x_0}f(x)\cdot\lim_{x\to x_0}g(x).$$

Corollary. A constant multiplier can be taken outside of the limit.

$$\lim_{x\to x_0}\big(cf(x)\big)=c\lim_{x\to x_0}f(x).$$

Theorem 1.3. The limit of the ratio of two functions is equal to the ratio of the limits of these functions if the limit of the denominator is different from 0.

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to x_0}f(x)}{\lim\limits_{x\to x_0}g(x)}.$$

Theorem 1.4. The limit of the positive function is not negative.

These statements are also true when x tends to ∞.

1.2.3. Special limits

If for applying basic theorems about function limits there are the expressions of the following forms: $\left[\frac{0}{0}\right]$, $\left[\frac{\infty}{\infty}\right]$, $\left[\infty-\infty\right]$, $\left[0\cdot\infty\right]$, $\left[1^{\infty}\right]$, $\left[0^{0}\right]$, $\left[\infty^{0}\right]$

(which are called *indeterminate forms*) then special methods are used to obtain the answer (so-called evaluating *the indeterminate forms*).

The following limits are used to solve the examples:

$$\lim_{x\to 0}\frac{\sin x}{x}=1; \tag{1.19}$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=\lim_{\alpha\to 0}(1+\alpha)^{1/\alpha}=e=2.71828..., \tag{1.20}$$

which are called *the first* and *second remarkable limits* respectively.

We are reminding (see Section 1.1.4) that the number e is the base of a natural logarithm.

Calculating limits, it is also useful to keep in mind the equations, which follow from (1.19) and (1.20):

$$\begin{aligned}&\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}; \quad \lim_{x\to 0}\frac{\log(1+x)}{x}=1;\\ &\lim_{x\to 0}\frac{a^x-1}{x}=\log a \quad (a>0); \quad \lim_{x\to 0}\frac{(1+x)^m-1}{x}=m.\end{aligned} \tag{1.21}$$

1.2.4. Examples of finding some limits

Example 1. Find $\lim\limits_{x\to 3}(x^2-7x+4)$.

Solution. Applying the theorems on limits (Theorem 1.1) and replacing x in the analytical expression with its limit value, we obtain

$$\lim_{x\to 3}(x^2-7x+4)=\lim_{x\to 3}x^2-7\lim_{x\to 3}x+4=9-21+4=-8.$$

Example 2. Find $\lim\limits_{x\to 3}\dfrac{x^2-9}{x^2-3x}$.

Solution. Numerator and denominator of the fraction tend to zero when x tends to 3 (it is usual to say that the indeterminate form of $\left[\dfrac{0}{0}\right]$). We have

$$\lim_{x\to 3}\frac{x^2-9}{x^2-3x}=\left[\frac{0}{0}\right]=\lim_{x\to 3}\frac{(x-3)(x+3)}{x(x-3)}.$$

Since, in the definition of the function limit, it is mentioned, that for finding the limit of a function, the values of the function at the limit point can be ignored, then $x - 3 \neq 0$. As a result, we can divide numerator by the denominator and obtain $\lim\limits_{x\to 3}\frac{x+3}{x}=\frac{3+3}{3}=2$.

Example 3. Find $\lim\limits_{x\to 0}\frac{\sqrt{x+4}-2}{x}$ (indeterminate form $\left[\frac{0}{0}\right]$).

Solution. Let us multiply numerator and denominator of the fraction by sum $\sqrt{x+4}+2$ (the conjugate). We use a well-known algebraic formula $a^2 - b^2 = (a - b)(a + b)$. We obtain

$$\lim_{x\to 0}\frac{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}{x(\sqrt{x+4}+2)}=\lim_{x\to 0}\frac{x+4-4}{x(\sqrt{x+4}+2)}=\lim_{x\to 0}\frac{1}{\sqrt{x+4}+2}=\frac{1}{4}.$$

Example 4. Find $\lim\limits_{x\to 3}\frac{\sqrt{3x+7}-\sqrt{2x+10}}{\sqrt{4x+13}-\sqrt{x+22}}$ (indeterminate form $\left[\frac{0}{0}\right]$).

Solution. Numerator and denominator of the fraction should be simultaneously multiplied by their conjugate, i.e. by the expression

$$\left(\sqrt{3x+7}+\sqrt{2x+10}\right)\left(\sqrt{4x+13}+\sqrt{x+22}\right).$$

We obtain

$$\lim_{x\to 3}\frac{\sqrt{3x+7}-\sqrt{2x+10}}{\sqrt{4x+13}-\sqrt{x+22}}=\lim_{x\to 3}\frac{\sqrt{3x+7}-\sqrt{2x+10}}{\sqrt{4x+13}-\sqrt{x+22}}\times$$
$$\times\frac{\left(\sqrt{3x+7}+\sqrt{2x+10}\right)\left(\sqrt{4x+13}+\sqrt{x+22}\right)}{\left(\sqrt{3x+7}+\sqrt{2x+10}\right)\left(\sqrt{4x+13}+\sqrt{x+22}\right)}=$$
$$=\lim_{x\to 3}\frac{\left(\sqrt{4x+13}+\sqrt{x+22}\right)}{3\left(\sqrt{3x+7}+\sqrt{2x+10}\right)}=\frac{5}{12}.$$

Example 5. Find $\lim\limits_{x\to 7}\frac{\sqrt[3]{x-6}-1}{x-7}$ (indeterminate form $\left[\frac{0}{0}\right]$).

Solution. Let us use a well-known algebraic formula

$$(a - b)(a^2 + ab + b^2) = a^3 - b^3.$$

Let $a=\sqrt[3]{x-6}$, $b = 1$. Therefore, in order to obtain a difference of cubes in the numerator, we should multiply it by $\left(\sqrt[3]{(x-6)^2}+\sqrt[3]{x-6}+1\right)$. After multiplying numerator and denominator by this value, we obtain

$$\lim_{x\to 7}\frac{\sqrt[3]{x-6}-1}{x-7}=\lim_{x\to 7}\frac{x-7}{(x-7)\left(\sqrt[3]{(x-6)^2}+\sqrt[3]{x-6}+1\right)}=$$

$$=\lim_{x\to 7}\frac{1}{\sqrt[3]{(x-6)^2}+\sqrt[3]{x-6}+1}=\frac{1}{1+1+1}=\frac{1}{3}.$$

Example 6. Find $\lim\limits_{x\to\infty}\dfrac{2x^3+x^2+5}{3x^3+x-1}$ (indeterminate form $\left[\dfrac{\infty}{\infty}\right]$).

Solution. We divide both numerator and denominator by the highest degree x found in the parts of the fraction, and then pass to the limit calculating:

$$\lim_{x\to\infty}\frac{2x^3+x^2+5}{3x^3+x-1}=\lim_{x\to\infty}\frac{2+1/x+5/x^3}{3+1/x^2-1/x^3}=\frac{\lim\limits_{x\to\infty}2+\lim\limits_{x\to\infty}\dfrac{1}{x}+\lim\limits_{x\to\infty}\dfrac{5}{x^3}}{\lim\limits_{x\to\infty}3+\lim\limits_{x\to\infty}\dfrac{1}{x^2}-\lim\limits_{x\to\infty}\dfrac{1}{x^3}}=\frac{2}{3},$$

since when $x\to\infty$ quantities $1/x$, $1/x^2$ и $1/x^3$ are infinitesimals, i.e. the limits of these quantities are equal to zero, when $x\to\infty$. Now it is possible to apply the theorem about the quotient limit.

Example 7. Find $\lim\limits_{x\to\infty}\left(\dfrac{x^3}{5x^2+1}-\dfrac{3x^2}{15x+1}\right)$ (indeterminate form $\left[\infty-\infty\right]$).

Solution. We bring the expression on the common denominator, and then divide numerator and denominator into the highest power of x found in the fraction.

$$\lim_{x\to\infty}\left(\frac{x^3}{5x^2+1}-\frac{3x^2}{15x+1}\right)=\lim_{x\to\infty}\frac{15x^4+x^3-15x^4-3x^2}{(5x^2+1)(15x+1)}=$$

$$=\lim_{x\to\infty}\frac{x^3-3x^2}{(5x^2+1)(15x+1)}=\lim_{x\to\infty}\frac{1-3/x}{\left(5+\dfrac{1}{x^2}\right)(15+1/x)}=1/75.$$

Example 8. Find $\lim\limits_{x\to\infty}\dfrac{3x^4-2}{\sqrt{x^8+3x+2}}$ (indeterminate form $\left[\dfrac{\infty}{\infty}\right]$).

Solution. We divide the numerator and denominator by the highest degree of x found in the example, i.e. by x^4.

$$\lim_{x\to\infty}\frac{3x^4-2}{\sqrt{x^8+3x+2}}=\lim_{x\to\infty}\frac{3-\dfrac{2}{x^4}}{\sqrt{1+\dfrac{3}{x^7}+\dfrac{4}{x^8}}}=\frac{3}{1}=3.$$

Example 9. Find $\lim\left(\sqrt{x^2+8x+3}-\sqrt{x^2+4x+3}\right)$ (indeterminate form $[\infty-\infty]$).

Solution. Multiply and divide the considered expression by its conjugate:

$$\lim_{x\to+\infty}\left(\sqrt{x^2+8x+3}-\sqrt{x^2+4x+3}\right)=$$

$$=\lim_{x\to+\infty}\left(\sqrt{x^2+8x+3}-\sqrt{x^2+4x+3}\right)\frac{\left(\sqrt{x^2+8x+3}+\sqrt{x^2+4x+3}\right)}{\sqrt{x^2+8x+3}+\sqrt{x^2+4x+3}}=$$

$$=\lim_{x\to+\infty}\frac{x^2+8x+3-x^2-4x-3}{\sqrt{x^2+8x+3}+\sqrt{x^2+4x+3}}=\lim_{x\to+\infty}\frac{4x}{\sqrt{x^2+8x+3}+\sqrt{x^2+4x+3}}=$$

$$=\lim_{x\to+\infty}\frac{4}{\sqrt{1+8/x+3/x^2}+\sqrt{1+4/x+3/x^2}}=\frac{4}{2}=2.$$

Example 10. Find $\lim\limits_{x\to0}\dfrac{\sin 4x}{x}$ (indeterminate form $\left[\dfrac{0}{0}\right]$).

Solution. We use the first remarkable limit:

$$\lim_{x\to0}\frac{\sin 4x}{x}=\lim_{y\to0}\frac{\sin 4x}{4x}\cdot4=4.$$

Example 11. Find $\lim\limits_{h\to0}\dfrac{\cos(x+h)-\cos(x-h)}{h}$ (indeterminate form $\left[\dfrac{0}{0}\right]$).

Solution. Using the formula

$$\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2},$$